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DEFORMATION OF A COMPOSITE ELASTIC PLANE WEAKENED BY A PERIODIC SYSTEM OF THE ARBITRARILY LOADED SLITS*

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The paper deals with the problems of periodic system of cuts distributed along the boundary of a bond connecting two elastic half-planes and acted upon by nonperiodic loads. In one problem it is assumed that the cuts are open, with normal and tangential stresses applied to their edges, while in another problem the edges touch each other and are loaded by tangential stresses. The method of solution is based on the simultaneous use of the discrete Fourier transformation and the theory of boundary value problems for automorphous analytic functions. The solutions are otained in quadratures. Other classes of problems to which the proposed methods can be applied, are described.

Generally speaking, in the case of irregular loads, the solution is usually based on the theory of representation of the symmetry groups /1,2/, and in the case of certain types of symmetry, particularly the translational, on the discrete Fourier transforms /3-6/. However the objects of transformation may be different in one and the same problem, and their choice affects significantly the solvability of the boundary value problem for the transformed quantities in the cell of periods. Below two problems of the theory of cracks are solved in quadratures to illustrate the effective simultaneous use of the discrete Fourier transformation and the Muskhelishvili method.

1. Let on elastic plane $z = x + iy$, obtained by bonding together two half-planes $y \geqslant 0$ and $y \leqslant 0$ with different elastic constants, be weakened along the line of bond by a 2π-periodic system of open slits. In the k-th band of periods $(2k-1)\pi \leqslant x \leqslant (2k+1)\pi$, $k = 0, \pm 1, \pm 2, \ldots$ the system L_k is composed of the segments of a straight line $a_n + 2k\pi < x < b_n + 2k\pi$, $y = 0$, $n = 1, 2, \ldots, N$. Arbitrary, nonperiodic, normal and tangential stresses are applied to the slit edges (the periodic problem was solved in /7/)

$$(\sigma_y - i\tau_{xy})(x \pm i0) = g_\pm(x), \quad x \in L_k \tag{1.1}$$

We require to find the elastic displacements of the plane $(u + iv)(z)$ corresponding to the case of finite energy of deformation near the points separating the boundary conditions.

Let us consider an auxilliary sequence of the boundary value problems in which the right-hand side of (1.1) assumes the following (nonperiodic) values depending on the parameter φ:

$$(\sigma_y - i\tau_{xy})(x + 2k\pi \pm i0) = g_\pm^s(x)\, e^{ik\varphi}, \quad x \in L_0 \tag{1.2}$$
$$g_\pm^s(x) = {}^1/_2\pi^{-1} g_\pm(x + 2s\pi), \; s = 0, \pm 1, \pm 2, \ldots, \; \varphi \in [0, 2\pi]$$

We shall seek the solution of these problems in the form /8/

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,[c_j K(z) + c_{j+2} M(z)] \tag{1.3}$$
$$\sigma_y - i\tau_{xy} = c_j\,[K(z) + (z - \bar z)\,\overline{K'(z)}] + \delta_j K(\bar z) + c_{j+2}\,[M(z) - M(\bar z) + (z - \bar z)\,\overline{M'(z)}]$$
$$2\mu_j \frac{\partial}{\partial x}(u + iv) = c_j\,[\varkappa_j K(z) - (z - \bar z)\,\overline{K'(z)}] - \delta_j K(\bar z) + c_{j+2}\,[\varkappa_j M(z) + M(\bar z) - (z - \bar z)\,\overline{M'(z)}]$$
$$\delta_1 = c_2, \quad \delta_2 = c_1, \quad c_1 = (\varkappa_1 + m)^{-1}$$
$$c_2 = (1 + m\varkappa_2)^{-1}, \quad c_3 = m\,(1 + \varkappa_2), \quad c_4 = 1 + \varkappa_1, \quad m = \mu_1\mu_2^{-1}$$

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$$\lim_{y\to 0} yK'(z) = 0, \quad \lim_{y\to 0} yM'(z) = 0$$

$$z \neq a_n + 2k\pi, \quad z \neq b_n + 2k\pi, \quad k = 0, \pm 1, \ldots$$

The functions $K(z)$ and $M(z)$ are regular in the complex z-plane except, perhaps, on the real axis; the index $j = 1$ ($j = 2$) denotes the parameters referring to the half-plane $y \geqslant 0$ ($y \leqslant 0$); for the plane deformation $\varkappa_j = 3 - 4\nu_j$, for the generalized plane state of stress $\varkappa_j = (3 - \nu_j)(1 + \nu_j)^{-1}$, μ_j is the shear modulus and ν_j is the Poisson's ratio. Let us put

$$K(z) = e^{i\alpha z} K_0(z), \quad M(z) = e^{i\alpha z} M_0(z), \quad \alpha = {}^1/_2\pi^{-1}\varphi \tag{1.4}$$

and substitute (1.4) into (1.3) and (1.2). We then find that the functions $K_0(z)$ and $M_0(z)$ should be piecewise regular and now 2π-periodic solutions in x ($x \in L_0$, $k = 0, \pm 1, \ldots$) of two problems

$$K_0^+(x + 2k\pi) + gK_0^-(x + 2k\pi) = e^{-i\alpha x} l^s(x) \tag{1.5}$$

$$M_0^+(x + 2k\pi) - M_0^-(x + 2k\pi) = e^{-i\alpha x} m^s(x)$$

$$g = c_1^{-1}c_2, \quad l^s(x) = \Delta c_1^{-1}[c_3 g_-^s(x) + c_4 g_+^s(x)]$$

$$m^s(x) = \Delta[g_+^s(x) - g_-^s(x)], \quad \Delta^{-1} = c_3 + c_4$$

Such solutions based on the theory of boundary value problems for the automorphous functions /9/ were constructed in /7/. Here however they assume a different form, because of the factorization of (1.4) and the conditions of decay of the functions $K(z)$ and $M(z)$ with $z \to \infty$. A solution of the problem, with a discontinuity, which vanishes as $z \to -i\infty$, is given by the automorphous analog of the Cauchy-type integral (/9/, Sect.52) with the basic automorphous function e^{iz} by:

$$M_0(z) = \frac{1}{2\pi} \int_{L_0} \frac{e^{it - i\alpha t} m^s(t)\, dt}{e^{it} - e^{iz}}$$

The canonical solution $X(z)$ of the Riemann problem (1.5) in the class of functions with integrable singularities at the points a_n and b_n, is based on the solution of the problem of discontinuity. Changing the form of the known method of approach (/9/, Sect.52) appropriately we obtain, in the case of open contours,

$$X(z) = \prod_{n=1}^{N} (e^{iz} - e^{ib_n}) e^{\Gamma(z)}, \quad \Gamma(z) = \frac{1}{2\pi} \int_{L_0} \frac{\ln(-g) e^{it}\, dt}{e^{it} - e^{iz}} = \left(\frac{1}{2} - i\gamma\right) \sum_{n=1}^{N} \ln \frac{e^{iz} - e^{ib_n}}{e^{iz} - e^{ia_n}}$$

where the principal value of the logarithm is used in computing $\gamma = {}^1/_2\pi^{-1}\ln g$. From the above relations follows

$$X(z) = \prod_{n=1}^{N} (e^{iz} - e^{ia_n})^{-1/2 + i\gamma} (e^{iz} - e^{ib_n})^{-1/2 - i\gamma}$$

and the branch of the canonical solution chosen here is determined by the asymptotics $e^{iNz}X(z) \to 1$, $z \to -i\infty$. The general solution of the Riemann problem vanishing at the lower end of the band of periods is found in the usual manner. Taking into account (1.4), we obtain

$$K(z) = X(z) e^{i\alpha z} \left[\frac{1}{2\pi} \int_{L_0} \frac{e^{it - i\alpha t} l^s(t)\, dt}{(e^{it} - e^{iz}) X^+(t)} + P_N^s(e^{iz})\right] \tag{1.6}$$

$$P_N^s(t) = \sum_{n=1}^{N} C_n^s t^{n-1}, \quad M(z) = \frac{1}{2\pi} \int_{L_0} \frac{\exp(it - i\alpha t + i\alpha z) m^s(t)\, dt}{e^{it} - e^{iz}}$$

The complex constants C_n^s are found from the conditions of uniqueness of the displacements during the passage around the cuts

$$\int_{a_n}^{b_n} \frac{\partial}{\partial x} [(u+iv)(x+i0)-(u+iv)(x-i0)]\,dx = 0, \quad n=1,\ldots,N \tag{1.7}$$

Let us return to the problem (1.1). The solution (1.3)—(1.7) of the problem (1.2) corresponds to the boundary conditions

$$\begin{aligned} \sigma_y^{s\varphi}(x+2k\pi \pm i0) &= \operatorname{Re} g_\pm^s(x)\cos k\varphi - \operatorname{Im} g_\pm^s(x)\sin k\varphi \\ \tau_{xy}^{s\varphi}(x+2k\pi \pm i0) &= -\operatorname{Re} g_\pm^s(x)\sin k\varphi - \operatorname{Im} g_\pm^s(x)\cos k\varphi \\ x \in L_0, \quad k &= 0, \pm 1, \ldots \end{aligned} \tag{1.8}$$

Integrating this solution and the conditions over the whole z-plane, in φ, from 0 to 2π, yields a new solution (a Green's function $u^s + iv^s$ of a kind) corresponding to the new conditions

$$\begin{aligned} {}^1/_2\pi^{-1}\sigma_y^s(x+2k\pi \pm i0) &= \operatorname{Re}\ g_\pm^s(x)\delta_{k0} \\ {}^1/_2\pi^{-1}\tau_{xy}^s(x+2k\pi \pm i0) &= -\operatorname{Im} g_\pm^s(x)\delta_{k0}, \quad x\in L_0, \quad k=0,\pm 1,\ldots \end{aligned} \tag{1.9}$$

where δ_{ks} is the Kronecker delta. Superposition of the solutions u^s and v^s displaced along the x-axis by $2\pi s$

$$u(z) = \sum_{s=-\infty}^{\infty} \int_0^{2\pi} u^{s\varphi}(z+2\pi s)\,d\varphi, \quad v(z) = \sum_{s=-\infty}^{\infty} \int_0^{2\pi} v^{s\varphi}(z+2\pi s)\,d\varphi \tag{1.10}$$

clearly represents the solution of the initial problem, satisfying the conditions (1.1) at the point of the z-plane at which the series (1.10) converge.

Example. Let $N=1$, $a_1=-a$, $b_1=a$, $0<a<\pi$ be a band of periods of weakening due to a single slit. In this case $P_1^s(e^{iz}) = C_1^s$, and according to (1.6) and (1.7) we have

$$X(z) = (e^{iz}-e^{-ia})^{-1/2+i\gamma}(e^{iz}-e^{ia})^{-1/2-i\gamma} \tag{1.11}$$

$$C_1^s = -\left\{\int_{-a}^{a}\int_{-a}^{a} \frac{e^{i\alpha x}X^+(x)\,e^{it-i\alpha t}l^s(t)\,dt\,dx}{(e^{it}-e^{ix})X^+(t)} + \frac{\pi\Delta}{c_1}\int_{-a}^{a}[(\varkappa_1-1)g_+^s(x) - m(\varkappa_2-1)g_-^s(x)]\,dx\right\}\left[2\pi\int_{-a}^{a}X^+(x)\,l^{i\alpha x}\,dx\right]^{-1}$$

The integrals appearing in this expression should be taken as their Cauchy principal values.

Let us find the asymptotics of the stresses growing without bounds at the points $z=\pm a + 2k\pi$, $k=0,\pm 1,\ldots$. Assuming that $K^+(x)=K^-(x)$ outside L_k, we obtain, by virtue of (1.3), for the case of an arbitrary N, $(\sigma_y - i\tau_{xy})(x) = (c_1+c_2)K(x)$. Integrating the function $K(x)$ with respect to φ from 0 to 2π and taking into account (1.9) we obtain, according to (1.4) and (1.6),

$$(\sigma_y - i\tau_{xy})^s(x) = (c_1+c_2)\frac{X(x)}{2\pi i}\left\{\int_{L_0}\frac{l^s(t)\,dt}{(t-x)X^+(t)} + i\int_0^{2\pi} e^{i\alpha x}p_N^s(e^{i\varphi})\,d\varphi\right\} \tag{1.12}$$

The expression within the curly brackets is bounded when $x=a_n+2k\pi$ and $x=b_n+2k\pi$; the character of the singularities at these points is determined by the behavior of the function $X(z)$ in (1.6), and has the form

$$(\sigma_y - i\tau_{xy})^s(x+2k\pi) = (M_{kn}^{\mp} - iN_{kn}^{\mp})\,|x-d_n|^{-1/2\pm i\gamma}, \quad x\in(-\pi,\pi) \tag{1.13}$$

where the upper signs are taken when $d_n=a_n$ and the lower ones when $d_n=b_n$.

Let us return to the case $N=1$ and use it to show the method of calculating the intensity coefficients at any N and n. We write the canonical solution (1.11) in the form

$$X(z) = \frac{1}{2i}\exp\left(a\gamma - \frac{iz}{2}\right)\left(\sin\frac{z+a}{2}\right)^{-1/2+i\gamma}\left(\sin\frac{z-a}{2}\right)^{-1/2-i\gamma} \tag{1.14}$$

Then for $x\in L_0$ we have, putting $1=e^{i\pi}$,

$$X^+(x) = -\frac{\sqrt{g}}{2}\exp\left(a\gamma - \frac{ix}{2}\right)\left(\sin\frac{x+a}{2}\right)^{-1/2+i\gamma}\left(\sin\frac{a-x}{2}\right)^{-1/2-i\gamma} \tag{1.15}$$

If $x \in (a+2k\pi,\ \pi+2k\pi)$, $k = 0, \pm 1, \ldots$, then $X(x)$ is given by the formula (1.14). Substituting (1.14) and (1.15) into (1.12) and taking into account (1.13), we obtain

$$M_{k1}^+ - iN_{k1}^+ = \frac{(c_1+c_2)(2\sin a)^{-1/2+i\gamma}}{\pi\exp(1/2\, ia)}\left\{\int_{L_0}\left(\sin\frac{a+t}{2}\right)^{1/2-i\gamma}\left(\sin\frac{a-t}{2}\right)^{1/2+i\gamma}\times\right.$$
$$\left.\frac{\exp(1/2\, at)\, t^s(t)\, dt}{\sqrt{g}\,(t-a-2k\pi)} - \frac{i}{a}\int_0^{2\pi}\exp(a\gamma + ia\alpha + ik\varphi)\, C_1^s\, d\varphi\right\}$$

Remarks. 1°. We can use the above method without any alterations to solve the analogous problems for a set L_k of perfectly rigid, thin and weakly bent inclusions of arbitrary shape distributed along the boundary separating different materials in a composite plane, when the condition $(u+iv)(x \pm i0) = g_{\pm}(x)$, $x \in L_k$ should hold instead of (1.1), as well as the problems of pressure exerted by a system L_k of stamps on a half-plane, the stamps fully coupled or touching its boundary without friction. Following /7,10/ we can reduce the same mixed problems for a multilayer strip and half-plane to the normal Poincaré — Koch systems.

2°. The conditions (1.2) correspond to (1.1) which have been directly transformed over the index k, with parameter φ. Indeed, let us introduce the following discrete Fourier transformation $f^*(\varphi)$ of the function $f(k)$:

$$f^*(\varphi) = \sum_{k=-\infty}^{\infty} f(k)\, e^{-ik\varphi}$$

and the vector $U(z) = (u, v, \sigma_x, \tau_{xy})(z)$. Then the condition

$$(\sigma_y - i\tau_{xy})(x + 2k\pi \pm i0) = g_{\pm}{}^s(x)\,\delta_{k0},\quad x \in L_0$$

which essentially coincides with (1.1) (according to (1.8)—(1.10) it defines the Green's function of the problem (1.1)) and condition of continuity

$$U(2k\pi + \pi - 0 + iy) = U(2k\pi + \pi + 0 + iy),\quad -\infty < y < \infty$$

having been subjected to discrete Fourier transformation, will yield the following condition at the boundaries of the elastic strip $-\pi \leqslant x \leqslant \pi$:

$$(\sigma_y - i\tau_{xy})^*(x \pm i0) \doteq g_{\pm}{}^s(x),\quad x \in L_0$$
$$U^*(\pi - 0 + iy) = e^{i\varphi}U^*(-\pi + 0 + iy),\quad -\infty < y < \infty$$

Clearly, the above boundary conditions determine the same problem for the transformed quantities, that conditions (1.2) do for the stresses themselves in the whole plane. The discrete Fourier transform could be applied, in its explicit form, to the boundary value problems of the type (1.5) (but not to the equations of the theory of elasticity as in /6/) which are obtained by substituting the solution (1.3) into the condition (1.1). The results obtained by both methods coincide.

2. In the field of shear and normal compressive stresses homogeneous at infinity, the load-free slits can become completely closed and yet act as the stress concentrators thus presenting a danger from the point of view of the theory of fracture. We shall consider the problem of Sect.1 for the closed slits when the conditions (1.1) at their edges are replaced by the conditions of continuity

$$\tau_{xy}(x \pm i0) = h_{\pm}(x),\quad v(x+i0) = v(x-i0),\quad \sigma_y(x+i0) = \sigma_y(x-i0) \tag{2.1}$$
$$x \in L_k,\quad k = 0, \pm 1, \pm 2, \ldots$$

In contrast to the problem for the open slits /7,8,11/, the above problem has not apparently been studied, neither in the case of periodic loads, nor in the case of a finite number of slits. Let $h_{\pm}(x)$ denote the periodic functions $h_{\pm}(x + 2k\pi) = h_{\pm}(x)$, $x \in L_0$, $k = 0, \pm 1, \ldots$ self-equilibrated within the set. Retaining the solution in the form (1.3) and substituting it into (2.1), we obtain the following boundary value problems for the functions $M(z)$ and $\Phi(z) = -i(c_1 + c_2)K(z)$, 2π-periodic, regular in the z-plane with cuts, and vanishing at infinity:

$$M^+(x) - M^-(x) = m(x), \quad x \in L_0 \tag{2.2}$$

$$\operatorname{Re} \Phi^+(x) = r^+(x), \quad x \in L_0 \tag{2.3}$$

$$\operatorname{Re} \Phi^-(x) = r^-(x), \quad x \in L_0, \quad m(x) = -i\Delta\,[h_+(x) - h_-(x)] \tag{2.4}$$

$$r^+(x) = c_2 q(x) + r(x), \quad r^-(x) = -c_1 q(x) + r(x), \quad r(x) = -\Delta\,[c_4 h_+(x) + c_3 h_-(x)]$$

$$q(x) = m\Delta\,(\varkappa_1\varkappa_2 - 1)[h_+(x) - h_-(x)]$$

The solution of the problem of discontinuity (2.2) is given by (1.6). The Dirichlet problems (2.3) and (2.4) can be reduced, according to the known results (/12/,Sect.91) to two Riemann boundary value problems

$$\Psi^+(x) + \Psi^-(x) = \psi(x), \quad x \in L_0 \tag{2.5}$$

$$\Omega^+(x) - \Omega^-(x) = \omega(x), \quad x \in L_0 \tag{2.6}$$

$$\psi(x) = r^+(x) + r^-(x), \quad \omega(x) = r^+(x) - r^-(x) \tag{2.7}$$

for the 2π-periodic functions $\Psi(z)$ and $\Omega(z)$ defined by the equation $\Phi(z) = \Psi(z) + \Omega(z)$ and satisfying the additional conditions

$$\bar{\Psi}(z) = \Psi(z), \quad \bar{\Omega}(z) = -\Omega(z) \tag{2.8}$$

The solutions of the problems (2.5) and (2.6) in the class of functions vanishing at infinity and integrable near the points $a_n, b_n, n = 1, 2, \ldots, N$, have the form

$$\Psi(z) = \frac{X(z)}{2\pi} \int_{L_0} \frac{e^{it}\psi(t)\,dt}{(e^{it} - e^{iz})\,X^+(t)} + P_N(e^{iz})\,X(z) \tag{2.9}$$

$$\Omega(z) = \frac{1}{2\pi} \int_{L_0} \frac{e^{it}\omega(t)\,dt}{e^{it} - e^{iz}}, \quad P_N(t) = \sum_{n=1}^{N} C_n t^{n-1}$$

$$X(z) = \prod_{n=1}^{N} (e^{iz} - e^{ia_n})^{-1/2} (e^{iz} - e^{ib_n})^{-1/2}$$

where C_n are complex constants determined by the first condition of (2.8) and the condition analogous to (1.7)

$$\int_{a_n}^{b_n} \frac{\partial}{\partial x}\,[u(x + i0) - u(x - i0)]\,dx = 0, \quad n = 1, 2, \ldots, N \tag{2.10}$$

In the case of a finite number of slits in the z-plane distributed over the segments L_0 only, the solution can be obtained by expanding the exponential functions (1.6) and (2.9) into series in powers of t and z /13/. Restricting ourselves to the first terms of the expressions we obtain, in the limit,

$$M(z) = \frac{1}{2\pi i} \int_{L_0} \frac{m(t)\,dt}{t - z}, \quad \Omega(z) = \frac{1}{2\pi i} \int_{L_0} \frac{\omega(t)\,dt}{t - z}$$

$$\Psi(z) = \frac{X(z)}{2\pi i} \int_{L_0} \frac{\psi(t)\,dt}{(t - z)\,X^+(t)} + X(z)\,P_N(z), \quad X(z) = \prod_{n=1}^{N} (z - a_n)^{-1/2} (z - b_n)^{-1/2}$$

where C_n are real constants satisfying the condition (2.10).

3. In the case of a homogeneous plane the problem (2.1) can be solved for the arbitrary nonperiodic loads. Indeed, in $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$, then the solution can be separated into a sum of a) symmetric with respect to the x-axis, and b) skew symmetric solutions. The problem a) represents a trivial fundamental problem for the lower half-plane $y \leqslant 0$ with the boundary conditions

$$\tau_{xy}(x - i0) = -{}^1/_2\,[h_+(x) - h_-(x)], \quad v(x - i0) = 0, \quad x \in (-\infty, \infty)$$

and problem b) is equivalent to the problem (1.1) in which we must put

$$\operatorname{Re} g_{\pm}(x) = 0, \quad \operatorname{Im} g_{\pm}(x) = -{}^1/_2\,[h_+(x) + h_-(x)]$$

The second condition of (2.1) will hold by virtue of symmetry. In the general case the problem (2.1) remains unsolved.

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